

Tripartite loss model for Mach-Zehnder interferometers with application to phase sensitivity : Complete expressions for measurement operator mean values, variances, and cross correlations

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Abstract

A generalized analytical tripartite loss model is posited for Mach-Zehnder interferometer (MZI) phase sensitivity which is valid for both arbitrary photon input states and arbitrary system environmental states. This model is shown to subsume the phase sensitivity models for the lossless MZI and the ground state MZI. It can be employed to develop specialized models useful for estimating phase sensitivities, as well as for performing associated design trade-off analyses, for MZIs which operate in environmental regimes that are not contained within the ground state MZI's envelope of validity. As a simple illustration of its utility, the model is used to develop phase sensitivity expressions for an MZI with "excited" internal arms and an MZI with "excited" output channels. These expressions yield a conditional relationship between the expected number of photons entering an MZI and its efficiency parameters which - when satisfied - predicts an enhanced phase sensitivity for the MZI with "excited" output channels relative to that for the MZI with "excited" internal arms.

I. INTRODUCTION

A loss model for Mach-Zehnder interferometers (MZIs) has recently been reported by us in the literature[1] (hereafter referred to as PI). This model is used to develop a phase sensitivity expression for arbitrary photon input states that is generally applicable for lossy MZIs employing subunit efficiency homodyne detection schemes. In particular, it is shown there that this phase sensitivity $\Delta^2\varphi$ is given by

$$\Delta^2\varphi = \frac{\left[\begin{aligned} &\kappa_\alpha^2 \Delta^2 C_\alpha + \kappa_\gamma^2 \Delta^2 C_\gamma + \kappa_\varepsilon^2 \Delta^2 C_\varepsilon + \kappa_{\alpha,\gamma}^2 \Delta^2 C_{\alpha,\gamma} + \kappa_{\alpha,\varepsilon}^2 \Delta^2 C_{\alpha,\varepsilon} \\ &+ \kappa_{\gamma,\varepsilon}^2 \Delta^2 C_{\gamma,\varepsilon} + \kappa_\alpha \kappa_{\alpha,\gamma} \langle \hat{D}_{\alpha;\alpha,\gamma} \rangle + \kappa_\alpha \kappa_{\alpha,\varepsilon} \langle \hat{D}_{\alpha;\alpha,\varepsilon} \rangle + \kappa_\gamma \kappa_{\alpha,\gamma} \langle \hat{D}_{\gamma;\alpha,\gamma} \rangle \\ &+ \kappa_\gamma \kappa_{\gamma,\varepsilon} \langle \hat{D}_{\gamma;\gamma,\varepsilon} \rangle + \kappa_\varepsilon \kappa_{\alpha,\varepsilon} \langle \hat{D}_{\varepsilon;\alpha,\varepsilon} \rangle + \kappa_\varepsilon \kappa_{\gamma,\varepsilon} \langle \hat{D}_{\varepsilon;\gamma,\varepsilon} \rangle + \kappa_{\alpha,\gamma} \kappa_{\alpha,\varepsilon} \langle \hat{D}_{\alpha,\gamma;\alpha,\varepsilon} \rangle \\ &+ \kappa_{\alpha,\gamma} \kappa_{\gamma,\varepsilon} \langle \hat{D}_{\alpha,\gamma;\gamma,\varepsilon} \rangle + \kappa_{\alpha,\varepsilon} \kappa_{\gamma,\varepsilon} \langle \hat{D}_{\alpha,\varepsilon;\gamma,\varepsilon} \rangle \end{aligned} \right]}{\left| \kappa_\alpha \frac{\partial \langle \hat{C}_\alpha \rangle}{\partial \varphi} + \kappa_{\alpha,\gamma} \frac{\partial \langle \hat{C}_{\alpha,\gamma} \rangle}{\partial \varphi} + \kappa_{\alpha,\varepsilon} \frac{\partial \langle \hat{C}_{\alpha,\varepsilon} \rangle}{\partial \varphi} \right|^2}, \quad (1)$$

where φ is the associated phase angle to be measured; the subscripts $x \in \{\alpha, \gamma, \varepsilon\}$ refer to the three regions used in the model ($0 < x \leq 1$ is the value of the efficiency parameter for region x); and the subscripted κ 's are constants which depend upon the regional efficiency parameters. The quantities $\Delta^2 C_x = \langle \hat{C}_x^2 \rangle - \langle \hat{C}_x \rangle^2$ and $\langle \hat{D}_{x;y} \rangle = \langle \hat{C}_x \hat{C}_y + \hat{C}_y \hat{C}_x \rangle - 2 \langle \hat{C}_x \rangle \langle \hat{C}_y \rangle$ are measurement operator variances and cross correlations (the operator \hat{C} measures the difference in the number of photons exiting the output ports), respectively, and $\langle \hat{Y} \rangle = \langle \Psi | \hat{Y} | \Psi \rangle$ is the mean value of the operator \hat{Y} for the system state $|\Psi\rangle = |\psi_{a_{in}, b_{in}}\rangle |\psi_c\rangle |\psi_d\rangle |\psi_e\rangle |\psi_f\rangle |\psi_g\rangle |\psi_h\rangle$. Here $|\psi_{a_{in}, b_{in}}\rangle$ is the preloss input state and $|\psi_x\rangle$, $x \in \{c, d, e, f, g, h\}$, is the normalized system environmental state associated with the system environmental annihilation operator \hat{x} . Precise definitions for the subscripted κ 's and \hat{C}_x operators appearing above are provided by eqs.(1) - (5) in PI. The reader is referred to section II in PI for a description of the physical assumptions and the regional architecture upon which the model is based.

The primary focus of PI was the development of the phase sensitivity expression for the ground state MZI. Such an MZI is defined in terms of our model as one which exhibits loss via non-unit regional efficiency parameter values and for which all system environmental states are vacuum states, i.e. $|\psi_x\rangle = |0\rangle$, $x \in \{c, d, e, f, g, h\}$, so that $|\Psi\rangle = |\psi_{a_{in}, b_{in}}\rangle |0\rangle |0\rangle |0\rangle |0\rangle |0\rangle |0\rangle$. In this case, the ground state phase sensitivity $\Delta^2\varphi_{gs}$ is given by eq.(1) with all but the first, fourth, and fifth terms in the numerator and the

first term in the denominator set equal to zero (see eqs.(11) and (12) in PI).

This expression for $\Delta^2\varphi_{gs}$ is a useful result because - as discussed in PI - it represents to a good approximation MZI phase sensitivity for a wide range of MZI environmental temperatures for frequencies in the near-IR to the near-UV region of the electromagnetic spectrum. However, for certain MZI applications it may be important to model the phase sensitivity for conditions where the system environmental states are not vacuum states, i.e. the MZI is not in its ground state, so that eq.(12) in PI does not apply. When using eq.(1) to model $\Delta^2\varphi$ for such cases, it is necessary to have the complete analytical expressions for each term appearing in the right hand side of this equation.

The purpose of this article is to extend the results of PI by providing the expressions for these terms so that eq.(1) is a complete model that may be more generally useful for MZI phase sensitivity analyses involving system environmental regimes which are not contained within the envelope of validity for the ground state model. Complete expressions for the partial phase derivatives of the measurement operator mean values, the measurement operator variances, and the measurement operator cross correlations appearing in the right hand side of eq.(1) are provided, respectively, in the following three sections of this paper. These expressions are validated in the final section of this paper by demonstrating that - when used in conjunction with eq.(1) - they yield the phase sensitivities for the lossless MZI and the ground state MZI previously developed in PI. As an additional illustration of its utility as an analytical tool, the model is employed to provide the phase sensitivity expressions for an MZI with "excited" internal arms, i.e. an "excited" γ -region configuration, and for an MZI with "excited" output channels, i.e. an "excited" ε -region configuration. These expressions are used to establish an associated conditional relationship between the expected number of photons entering an MZI and its regional efficiency parameter values. When this condition is satisfied, then the phase sensitivity of the "excited" γ -region configuration is more degraded than that of the "excited" ε -region configuration.

II. PARTIAL PHASE DERIVATIVES OF MEASUREMENT OPERATOR MEAN VALUES

The expressions for the measurement operator mean values needed to evaluate the partial derivatives appearing in the denominator of eq.(1) are

$$\langle \hat{C}_\alpha \rangle = 4 \left\{ \left(\langle \hat{\rho}_b^\dagger \hat{\rho}_b \rangle - \langle \hat{\rho}_a^\dagger \hat{\rho}_a \rangle \right) \cos \varphi - \left(\langle \hat{\rho}_a^\dagger \hat{\rho}_b \rangle + \langle \hat{\rho}_b^\dagger \hat{\rho}_a \rangle \right) \sin \varphi \right\}, \quad (2)$$

$$\langle \hat{C}_{\alpha,\gamma} \rangle = 2 \left\{ \begin{aligned} & \left(\langle \hat{\rho}_b^\dagger \hat{f} \rangle + \langle \hat{f}^\dagger \hat{\rho}_b \rangle \right) + i \left(\langle \hat{\rho}_a^\dagger \hat{f} \rangle - \langle \hat{f}^\dagger \hat{\rho}_a \rangle \right) \\ & - i \left(\langle \hat{\rho}_b^\dagger \hat{e} \rangle e^{-i\varphi} - \langle \hat{e}^\dagger \hat{\rho}_b \rangle e^{i\varphi} \right) - \left(\langle \hat{\rho}_a^\dagger \hat{e} \rangle e^{-i\varphi} + \langle \hat{e}^\dagger \hat{\rho}_a \rangle e^{i\varphi} \right) \end{aligned} \right\}, \quad (3)$$

and

$$\begin{aligned} \langle \hat{C}_{\alpha,\varepsilon} \rangle &= \{ \langle \hat{\rho}_a^\dagger \hat{g} \rangle (1 - e^{-i\varphi}) + \langle \hat{g}^\dagger \hat{\rho}_a \rangle (1 - e^{i\varphi}) \} + \{ \langle \hat{\rho}_b^\dagger \hat{h} \rangle (1 - e^{-i\varphi}) + \langle \hat{h}^\dagger \hat{\rho}_b \rangle (1 - e^{i\varphi}) \} \\ &+ i \{ \langle \hat{\rho}_a^\dagger \hat{h} \rangle (1 + e^{-i\varphi}) - \langle \hat{h}^\dagger \hat{\rho}_a \rangle (1 + e^{i\varphi}) \} - i \{ \langle \hat{\rho}_b^\dagger \hat{g} \rangle (1 + e^{-i\varphi}) - \langle \hat{g}^\dagger \hat{\rho}_b \rangle (1 + e^{i\varphi}) \}, \end{aligned} \quad (4)$$

where

$$\hat{\rho}_a = \sqrt{\alpha} \hat{a}_{in} + \sqrt{1-\alpha} \hat{c}$$

and

$$\hat{\rho}_b = \sqrt{\alpha} \hat{b}_{in} + \sqrt{1-\alpha} \hat{d}.$$

Here $0 < \alpha \leq 1$ is the efficiency parameter for the α region of the model, \hat{a}_{in} and \hat{b}_{in} are the input port annihilation operators, \hat{c} and \hat{d} are the α region environmental annihilation operators, \hat{e} and \hat{f} are the γ region environmental annihilation operators, and \hat{g} and \hat{h} are the ε region environmental annihilation operators.

The partial derivatives appearing in eq.(1) are readily obtained as follows from eqs. (2) - (4) :

$$\frac{\partial \langle \hat{C}_\alpha \rangle}{\partial \varphi} = -4 \{ \left(\langle \hat{\rho}_b^\dagger \hat{\rho}_b \rangle - \langle \hat{\rho}_a^\dagger \hat{\rho}_a \rangle \right) \sin \varphi + \left(\langle \hat{\rho}_a^\dagger \hat{\rho}_b \rangle + \langle \hat{\rho}_b^\dagger \hat{\rho}_a \rangle \right) \cos \varphi \}, \quad (5)$$

$$\frac{\partial \langle \hat{C}_{\alpha,\gamma} \rangle}{\partial \varphi} = -2 \left\{ \left(\langle \hat{\rho}_b^\dagger \hat{e} \rangle - i \langle \hat{\rho}_a^\dagger \hat{e} \rangle \right) e^{-i\varphi} + \left(\langle \hat{e}^\dagger \hat{\rho}_b \rangle + i \langle \hat{e}^\dagger \hat{\rho}_a \rangle \right) e^{i\varphi} \right\},$$

and

$$\frac{\partial \langle \hat{C}_{\alpha,\varepsilon} \rangle}{\partial \varphi} = \left\{ \begin{aligned} & \left[\left(\langle \hat{\rho}_a^\dagger \hat{h} \rangle - \langle \hat{\rho}_b^\dagger \hat{g} \rangle \right) + i \left(\langle \hat{\rho}_a^\dagger \hat{g} \rangle + \langle \hat{\rho}_b^\dagger \hat{h} \rangle \right) \right] e^{-i\varphi} \\ & + \left[\left(\langle \hat{h}^\dagger \hat{\rho}_a \rangle - \langle \hat{g}^\dagger \hat{\rho}_b \rangle \right) - i \left(\langle \hat{g}^\dagger \hat{\rho}_a \rangle + \langle \hat{h}^\dagger \hat{\rho}_b \rangle \right) \right] e^{i\varphi} \end{aligned} \right\}.$$

III. MEASUREMENT OPERATOR VARIANCES

Expressions for the measurement operator variances needed for the evaluation of the numerator of eq.(1) are

$$\Delta^2 C_\alpha = 16 \left[\begin{aligned} & \left(\left\langle \left(\hat{\rho}_b^\dagger \hat{\rho}_b - \hat{\rho}_a^\dagger \hat{\rho}_a \right)^2 \right\rangle - \left\langle \hat{\rho}_b^\dagger \hat{\rho}_b - \hat{\rho}_a^\dagger \hat{\rho}_a \right\rangle^2 \right) \cos^2 \varphi + \\ & \left(\left\langle \left(\hat{\rho}_a^\dagger \hat{\rho}_b + \hat{\rho}_b^\dagger \hat{\rho}_a \right)^2 \right\rangle - \left\langle \hat{\rho}_a^\dagger \hat{\rho}_b + \hat{\rho}_b^\dagger \hat{\rho}_a \right\rangle^2 \right) \sin^2 \varphi - \\ & \left(\left\langle \left\{ \left(\hat{\rho}_b^\dagger \hat{\rho}_b - \hat{\rho}_a^\dagger \hat{\rho}_a \right), \left(\hat{\rho}_a^\dagger \hat{\rho}_b + \hat{\rho}_b^\dagger \hat{\rho}_a \right) \right\} \right\rangle \right. \\ & \quad \left. - 2 \left\langle \hat{\rho}_b^\dagger \hat{\rho}_b - \hat{\rho}_a^\dagger \hat{\rho}_a \right\rangle \left\langle \hat{\rho}_a^\dagger \hat{\rho}_b + \hat{\rho}_b^\dagger \hat{\rho}_a \right\rangle \right) \sin \varphi \cos \varphi \end{aligned} \right], \quad (6)$$

where the term enclosed in curly braces is the anti-commutator defined by $\{\hat{X}, \hat{Y}\} = \hat{X}\hat{Y} + \hat{Y}\hat{X}$;

$$\Delta^2 C_\gamma = 4 \left[\begin{aligned} & \left\langle \hat{e}^\dagger \hat{e} \hat{f} \hat{f}^\dagger \right\rangle + \left\langle \hat{e} \hat{e}^\dagger \hat{f}^\dagger \hat{f} \right\rangle + \left\langle \hat{e}^\dagger \hat{f} \right\rangle^2 + \left\langle \hat{e} \hat{f}^\dagger \right\rangle^2 \\ & - \left\langle \hat{e}^\dagger \hat{e}^\dagger \hat{f} \hat{f} \right\rangle - \left\langle \hat{e} \hat{e} \hat{f}^\dagger \hat{f}^\dagger \right\rangle - 2 \left\langle \hat{e}^\dagger \hat{f} \right\rangle \left\langle \hat{e} \hat{f}^\dagger \right\rangle \end{aligned} \right]; \quad (7)$$

$$\Delta^2 C_\varepsilon = \left[\begin{aligned} & \left(\left\langle \hat{g}^\dagger \hat{g} \hat{g}^\dagger \hat{g} \right\rangle - \left\langle \hat{g}^\dagger \hat{g} \right\rangle^2 \right) + \left(\left\langle \hat{h}^\dagger \hat{h} \hat{h}^\dagger \hat{h} \right\rangle - \left\langle \hat{h}^\dagger \hat{h} \right\rangle^2 \right) \\ & - 2 \left(\left\langle \hat{g}^\dagger \hat{g} \hat{h}^\dagger \hat{h} \right\rangle - \left\langle \hat{g}^\dagger \hat{g} \right\rangle \left\langle \hat{h}^\dagger \hat{h} \right\rangle \right) \end{aligned} \right];$$

$$\Delta^2 C_{\alpha, \gamma} = 4 \left[\begin{aligned} & \left\langle \hat{F}_1 \hat{e} \hat{e} \right\rangle + \left\langle \hat{F}_1^\dagger \hat{e}^\dagger \hat{e}^\dagger \right\rangle + \left\langle \hat{F}_2 \hat{e}^\dagger \hat{e} \right\rangle + \left\langle \hat{F}_3 \hat{e} \hat{e}^\dagger \right\rangle - \left\langle \hat{F}_4 \hat{f} \hat{f} \right\rangle - \left\langle \hat{F}_4^\dagger \hat{f}^\dagger \hat{f}^\dagger \right\rangle + \\ & \left\langle \hat{F}_5 \hat{f}^\dagger \hat{f} \right\rangle + \left\langle \hat{F}_6 \hat{f} \hat{f}^\dagger \right\rangle + \left\langle \hat{F}_7 \hat{e} \hat{f} \right\rangle + \left\langle \hat{F}_7^\dagger \hat{e}^\dagger \hat{f}^\dagger \right\rangle - \left\langle \hat{F}_8 \hat{e}^\dagger \hat{f} \right\rangle - \left\langle \hat{F}_8^\dagger \hat{e} \hat{f}^\dagger \right\rangle - \\ & \left\langle \hat{\rho}_b^\dagger \hat{f} \right\rangle^2 - \left\langle \hat{\rho}_b \hat{f}^\dagger \right\rangle^2 + \left\langle \hat{\rho}_a^\dagger \hat{f} \right\rangle^2 + \left\langle \hat{\rho}_a \hat{f}^\dagger \right\rangle^2 - \\ & 2 \left(\left\langle \hat{\rho}_b^\dagger \hat{f} \right\rangle \left\langle \hat{\rho}_b \hat{f}^\dagger \right\rangle + \left\langle \hat{\rho}_a^\dagger \hat{f} \right\rangle \left\langle \hat{\rho}_a \hat{f}^\dagger \right\rangle + \left\langle \hat{\rho}_b^\dagger \hat{e} \right\rangle \left\langle \hat{\rho}_b \hat{e}^\dagger \right\rangle + \left\langle \hat{\rho}_a^\dagger \hat{e} \right\rangle \left\langle \hat{\rho}_a \hat{e}^\dagger \right\rangle \right) - \\ & \left(\left\langle \hat{\rho}_a^\dagger \hat{e} \right\rangle^2 - \left\langle \hat{\rho}_b^\dagger \hat{e} \right\rangle^2 \right) e^{-2i\varphi} - \left(\left\langle \hat{\rho}_a \hat{e}^\dagger \right\rangle^2 - \left\langle \hat{\rho}_b \hat{e}^\dagger \right\rangle^2 \right) e^{2i\varphi} - \\ & 2 \left(\left\langle \hat{\rho}_a^\dagger \hat{f} \right\rangle \left\langle \hat{\rho}_b^\dagger \hat{e} \right\rangle - \left\langle \hat{\rho}_a \hat{f}^\dagger \right\rangle \left\langle \hat{\rho}_b \hat{e} \right\rangle - \left\langle \hat{\rho}_b^\dagger \hat{f} \right\rangle \left\langle \hat{\rho}_a^\dagger \hat{e} \right\rangle - \left\langle \hat{\rho}_b \hat{f}^\dagger \right\rangle \left\langle \hat{\rho}_a \hat{e}^\dagger \right\rangle \right) e^{-i\varphi} - \\ & 2 \left(\left\langle \hat{\rho}_a \hat{f}^\dagger \right\rangle \left\langle \hat{\rho}_b \hat{e}^\dagger \right\rangle - \left\langle \hat{\rho}_a^\dagger \hat{f} \right\rangle \left\langle \hat{\rho}_b \hat{e} \right\rangle - \left\langle \hat{\rho}_b^\dagger \hat{f} \right\rangle \left\langle \hat{\rho}_a \hat{e}^\dagger \right\rangle - \left\langle \hat{\rho}_b \hat{f}^\dagger \right\rangle \left\langle \hat{\rho}_a \hat{e} \right\rangle \right) e^{i\varphi} - \\ & 2i \left[\left\langle \hat{\rho}_a^\dagger \hat{f} \right\rangle \left\langle \hat{\rho}_b^\dagger \hat{f} \right\rangle + \left\langle \hat{\rho}_a^\dagger \hat{f} \right\rangle \left\langle \hat{\rho}_b \hat{f}^\dagger \right\rangle + \left\langle \hat{\rho}_a \hat{e}^\dagger \right\rangle \left\langle \hat{\rho}_b^\dagger \hat{e} \right\rangle - \left\langle \hat{\rho}_a \hat{f}^\dagger \right\rangle \left\langle \hat{\rho}_b \hat{f}^\dagger \right\rangle - \right. \\ & \left. \left\langle \hat{\rho}_a \hat{f}^\dagger \right\rangle \left\langle \hat{\rho}_b^\dagger \hat{f} \right\rangle - \left\langle \hat{\rho}_a^\dagger \hat{e} \right\rangle \left\langle \hat{\rho}_b \hat{e}^\dagger \right\rangle + \left\langle \hat{\rho}_a^\dagger \hat{e} \right\rangle \left\langle \hat{\rho}_b^\dagger \hat{e} \right\rangle e^{-2i\varphi} - \left\langle \hat{\rho}_a \hat{e}^\dagger \right\rangle \left\langle \hat{\rho}_b \hat{e}^\dagger \right\rangle e^{2i\varphi} + \right. \\ & \left(\left\langle \hat{\rho}_a \hat{f}^\dagger \right\rangle \left\langle \hat{\rho}_a^\dagger \hat{e} \right\rangle - \left\langle \hat{\rho}_a^\dagger \hat{f} \right\rangle \left\langle \hat{\rho}_a^\dagger \hat{e} \right\rangle - \left\langle \hat{\rho}_b^\dagger \hat{f} \right\rangle \left\langle \hat{\rho}_b^\dagger \hat{e} \right\rangle - \left\langle \hat{\rho}_b \hat{f}^\dagger \right\rangle \left\langle \hat{\rho}_b^\dagger \hat{e} \right\rangle \right) e^{-i\varphi} + \\ & \left(\left\langle \hat{\rho}_b \hat{f}^\dagger \right\rangle \left\langle \hat{\rho}_b \hat{e}^\dagger \right\rangle + \left\langle \hat{\rho}_b^\dagger \hat{f} \right\rangle \left\langle \hat{\rho}_b \hat{e}^\dagger \right\rangle + \left\langle \hat{\rho}_a \hat{f}^\dagger \right\rangle \left\langle \hat{\rho}_a \hat{e}^\dagger \right\rangle - \left\langle \hat{\rho}_a^\dagger \hat{f} \right\rangle \left\langle \hat{\rho}_a \hat{e}^\dagger \right\rangle \right) e^{i\varphi} \end{aligned} \right], \quad (8)$$

where

$$\hat{F}_1 = \left(\hat{\rho}_a^\dagger \hat{\rho}_a^\dagger - \hat{\rho}_b^\dagger \hat{\rho}_b^\dagger + 2i \hat{\rho}_a^\dagger \hat{\rho}_b^\dagger \right) e^{-2i\varphi},$$

$$\begin{aligned}
\hat{F}_2 &= \hat{\rho}_a \hat{\rho}_a^\dagger + \hat{\rho}_b \hat{\rho}_b^\dagger + i \left(\hat{\rho}_a \hat{\rho}_b^\dagger - \hat{\rho}_a^\dagger \hat{\rho}_b \right), \\
\hat{F}_3 &= \hat{\rho}_a^\dagger \hat{\rho}_a + \hat{\rho}_b^\dagger \hat{\rho}_b + i \left(\hat{\rho}_a \hat{\rho}_b^\dagger - \hat{\rho}_a^\dagger \hat{\rho}_b \right), \\
\hat{F}_4 &= \hat{\rho}_a^\dagger \hat{\rho}_a^\dagger - \hat{\rho}_b^\dagger \hat{\rho}_b^\dagger - 2i \hat{\rho}_a^\dagger \hat{\rho}_b^\dagger, \\
\hat{F}_5 &= \hat{\rho}_a \hat{\rho}_a^\dagger + \hat{\rho}_b \hat{\rho}_b^\dagger + i \left(\hat{\rho}_a^\dagger \hat{\rho}_b - \hat{\rho}_b^\dagger \hat{\rho}_a \right), \\
\hat{F}_6 &= \hat{\rho}_a^\dagger \hat{\rho}_a + \hat{\rho}_b^\dagger \hat{\rho}_b + i \left(\hat{\rho}_a^\dagger \hat{\rho}_b - \hat{\rho}_b^\dagger \hat{\rho}_a \right), \\
\hat{F}_7 &= -2i \left(\hat{\rho}_a^\dagger \hat{\rho}_a^\dagger + \hat{\rho}_b^\dagger \hat{\rho}_b^\dagger \right) e^{-i\varphi}, \\
\hat{F}_8 &= \left[2 \left(\hat{\rho}_a \hat{\rho}_b^\dagger + \hat{\rho}_a^\dagger \hat{\rho}_b \right) + i \left(\hat{\rho}_a^\dagger \hat{\rho}_a + \hat{\rho}_a \hat{\rho}_a^\dagger - \hat{\rho}_b^\dagger \hat{\rho}_b - \hat{\rho}_b \hat{\rho}_b^\dagger \right) \right] e^{i\varphi};
\end{aligned}$$

$$\Delta^2 C_{\alpha, \varepsilon} = \left[\begin{aligned}
& \langle \hat{R}_1 \hat{g} \hat{g} \rangle + \langle \hat{R}_1^\dagger \hat{g}^\dagger \hat{g}^\dagger \rangle + \langle \hat{R}_2 \hat{g}^\dagger \hat{g} \rangle + \langle \hat{R}_3 \hat{g} \hat{g}^\dagger \rangle + \langle \hat{R}_4 \hat{h} \hat{h} \rangle + \langle \hat{R}_4^\dagger \hat{h}^\dagger \hat{h}^\dagger \rangle + \\
& \langle \hat{R}_5 \hat{h}^\dagger \hat{h} \rangle + \langle \hat{R}_6 \hat{h} \hat{h}^\dagger \rangle + \langle \hat{R}_7 \hat{g} \hat{h} \rangle + \langle \hat{R}_7^\dagger \hat{g}^\dagger \hat{h}^\dagger \rangle + \langle \hat{R}_8 \hat{g}^\dagger \hat{h} \rangle + \langle \hat{R}_8^\dagger \hat{g} \hat{h}^\dagger \rangle - \\
& \left(\langle \hat{\rho}_a^\dagger \hat{g} \rangle^2 + \langle \hat{\rho}_b^\dagger \hat{h} \rangle^2 + 2 \langle \hat{\rho}_a^\dagger \hat{g} \rangle \langle \hat{\rho}_b^\dagger \hat{h} \rangle \right) (1 - e^{-i\varphi})^2 - \\
& \left(\langle \hat{\rho}_a \hat{g}^\dagger \rangle^2 + \langle \hat{\rho}_b \hat{h}^\dagger \rangle^2 + 2 \langle \hat{\rho}_a \hat{g}^\dagger \rangle \langle \hat{\rho}_b \hat{h}^\dagger \rangle \right) (1 - e^{i\varphi})^2 + \\
& \left(\langle \hat{\rho}_a^\dagger \hat{h} \rangle^2 + \langle \hat{\rho}_b^\dagger \hat{g} \rangle^2 - 2 \langle \hat{\rho}_a^\dagger \hat{h} \rangle \langle \hat{\rho}_b^\dagger \hat{g} \rangle \right) (1 + e^{-i\varphi})^2 + \\
& \left(\langle \hat{\rho}_a \hat{h}^\dagger \rangle^2 + \langle \hat{\rho}_b \hat{g}^\dagger \rangle^2 - 2 \langle \hat{\rho}_a \hat{h}^\dagger \rangle \langle \hat{\rho}_b \hat{g}^\dagger \rangle \right) (1 + e^{i\varphi})^2 - \\
& 2 \left(\langle \hat{\rho}_a^\dagger \hat{g} \rangle \langle \hat{\rho}_a \hat{g}^\dagger \rangle + \langle \hat{\rho}_b^\dagger \hat{h} \rangle \langle \hat{\rho}_b \hat{h}^\dagger \rangle \right) (1 - e^{i\varphi}) (1 - e^{-i\varphi}) - \\
& 2 \left(\langle \hat{\rho}_a^\dagger \hat{h} \rangle \langle \hat{\rho}_a \hat{h}^\dagger \rangle + \langle \hat{\rho}_b^\dagger \hat{g} \rangle \langle \hat{\rho}_b \hat{g}^\dagger \rangle \right) (1 + e^{i\varphi}) (1 + e^{-i\varphi}) - \\
& 2i \left[\left(\langle \hat{\rho}_a^\dagger \hat{g} \rangle \langle \hat{\rho}_a^\dagger \hat{h} \rangle + \langle \hat{\rho}_b^\dagger \hat{h} \rangle \langle \hat{\rho}_a^\dagger \hat{h} \rangle \right) (1 - e^{-i\varphi}) (1 + e^{-i\varphi}) + \right. \\
& \left(\langle \hat{\rho}_a \hat{g}^\dagger \rangle \langle \hat{\rho}_b \hat{g}^\dagger \rangle + \langle \hat{\rho}_b \hat{h}^\dagger \rangle \langle \hat{\rho}_b \hat{g}^\dagger \rangle \right) (1 - e^{i\varphi}) (1 + e^{i\varphi}) + \\
& \left(\langle \hat{\rho}_a^\dagger \hat{g} \rangle \langle \hat{\rho}_b \hat{g}^\dagger \rangle + \langle \hat{\rho}_b^\dagger \hat{h} \rangle \langle \hat{\rho}_b \hat{g}^\dagger \rangle \right) (1 - e^{-i\varphi}) (1 + e^{i\varphi}) + \\
& \left. \left(\langle \hat{\rho}_a \hat{g}^\dagger \rangle \langle \hat{\rho}_a^\dagger \hat{h} \rangle + \langle \hat{\rho}_b \hat{h}^\dagger \rangle \langle \hat{\rho}_a^\dagger \hat{h} \rangle \right) (1 - e^{i\varphi}) (1 + e^{-i\varphi}) \right]
\end{aligned} \right], \quad (9)$$

where

$$\hat{R}_1 = \hat{\rho}_a^\dagger \hat{\rho}_a^\dagger (1 - e^{-i\varphi})^2 - \hat{\rho}_b^\dagger \hat{\rho}_b^\dagger (1 + e^{-i\varphi})^2 - 2i \hat{\rho}_a^\dagger \hat{\rho}_b^\dagger (1 - e^{-i\varphi}) (1 + e^{-i\varphi}),$$

$$\begin{aligned}
\hat{R}_2 &= \begin{bmatrix} \hat{\rho}_a \hat{\rho}_a^\dagger (1 - e^{i\varphi}) (1 - e^{-i\varphi}) + \hat{\rho}_b \hat{\rho}_b^\dagger (1 + e^{-i\varphi}) (1 + e^{i\varphi}) \\ -i \hat{\rho}_a \hat{\rho}_b^\dagger (1 - e^{i\varphi}) (1 + e^{-i\varphi}) + i \hat{\rho}_a^\dagger \hat{\rho}_b (1 + e^{i\varphi}) (1 - e^{-i\varphi}) \end{bmatrix}, \\
\hat{R}_3 &= \begin{bmatrix} \hat{\rho}_a^\dagger \hat{\rho}_a (1 - e^{-i\varphi}) (1 - e^{i\varphi}) + \hat{\rho}_b^\dagger \hat{\rho}_b (1 + e^{-i\varphi}) (1 + e^{i\varphi}) \\ -i \hat{\rho}_a \hat{\rho}_b^\dagger (1 + e^{-i\varphi}) (1 - e^{i\varphi}) + i \hat{\rho}_a^\dagger \hat{\rho}_b (1 - e^{-i\varphi}) (1 + e^{i\varphi}) \end{bmatrix}, \\
\hat{R}_4 &= \hat{\rho}_b^\dagger \hat{\rho}_b^\dagger (1 - e^{-i\varphi})^2 - \hat{\rho}_a^\dagger \hat{\rho}_a^\dagger (1 + e^{-i\varphi})^2 + 2i \hat{\rho}_a^\dagger \hat{\rho}_b^\dagger (1 - e^{-i\varphi}) (1 + e^{-i\varphi}), \\
\hat{R}_5 &= \begin{bmatrix} \hat{\rho}_b \hat{\rho}_b^\dagger (1 - e^{i\varphi}) (1 - e^{-i\varphi}) + \hat{\rho}_a \hat{\rho}_a^\dagger (1 + e^{i\varphi}) (1 + e^{-i\varphi}) \\ +i \hat{\rho}_a^\dagger \hat{\rho}_b (1 - e^{i\varphi}) (1 + e^{-i\varphi}) - i \hat{\rho}_a \hat{\rho}_b^\dagger (1 + e^{i\varphi}) (1 - e^{-i\varphi}) \end{bmatrix}, \\
\hat{R}_6 &= \begin{bmatrix} \hat{\rho}_b^\dagger \hat{\rho}_b (1 - e^{-i\varphi}) (1 - e^{i\varphi}) + \hat{\rho}_a^\dagger \hat{\rho}_a (1 + e^{-i\varphi}) (1 + e^{i\varphi}) \\ +i \hat{\rho}_a^\dagger \hat{\rho}_b (1 + e^{-i\varphi}) (1 - e^{i\varphi}) - i \hat{\rho}_a \hat{\rho}_b^\dagger (1 - e^{-i\varphi}) (1 + e^{i\varphi}) \end{bmatrix}, \\
\hat{R}_7 &= 2 \begin{bmatrix} \hat{\rho}_a^\dagger \hat{\rho}_b^\dagger (1 - e^{-i\varphi})^2 + \hat{\rho}_a^\dagger \hat{\rho}_b^\dagger (1 + e^{-i\varphi})^2 \\ +i \hat{\rho}_a^\dagger \hat{\rho}_a^\dagger (1 - e^{-i\varphi}) (1 + e^{-i\varphi}) - i \hat{\rho}_b^\dagger \hat{\rho}_b^\dagger (1 - e^{-i\varphi}) (1 + e^{-i\varphi}) \end{bmatrix}, \\
\hat{R}_8 &= \begin{bmatrix} 2 \left[\hat{\rho}_a \hat{\rho}_b^\dagger (1 - e^{i\varphi}) (1 - e^{-i\varphi}) - \hat{\rho}_a^\dagger \hat{\rho}_b (1 + e^{-i\varphi}) (1 + e^{i\varphi}) \right] \\ +i \left[(\hat{\rho}_a \hat{\rho}_a^\dagger + \hat{\rho}_a^\dagger \hat{\rho}_a) (1 + e^{-i\varphi}) (1 - e^{i\varphi}) + (\hat{\rho}_b \hat{\rho}_b^\dagger + \hat{\rho}_b^\dagger \hat{\rho}_b) (1 - e^{-i\varphi}) (1 + e^{i\varphi}) \right] \end{bmatrix};
\end{aligned}$$

and

$$\Delta^2 C_{\gamma, \varepsilon} = \begin{bmatrix} \begin{aligned} &\langle \hat{Q}_1 \hat{e} \hat{e} \rangle + \langle \hat{Q}_1^\dagger \hat{e}^\dagger \hat{e}^\dagger \rangle + \langle \hat{Q}_2 \hat{e}^\dagger \hat{e} \rangle + \langle \hat{Q}_3 \hat{e} \hat{e}^\dagger \rangle \\ &+ \langle \hat{Q}_4 \hat{f} \hat{f} \rangle + \langle \hat{Q}_4^\dagger \hat{f}^\dagger \hat{f}^\dagger \rangle + \langle \hat{Q}_5 \hat{f}^\dagger \hat{f} \rangle + \langle \hat{Q}_6 \hat{f} \hat{f}^\dagger \rangle \\ &+ \langle \hat{Q}_7 \hat{e} \hat{f} \rangle + \langle \hat{Q}_7^\dagger \hat{e}^\dagger \hat{f}^\dagger \rangle - \langle \hat{Q}_8 \hat{e}^\dagger \hat{f} \rangle - \langle \hat{Q}_8^\dagger \hat{e} \hat{f}^\dagger \rangle \\ &- \langle \hat{e}^\dagger \hat{g} \rangle^2 - \langle \hat{e} \hat{g}^\dagger \rangle^2 - \langle \hat{f}^\dagger \hat{h} \rangle^2 - \langle \hat{f} \hat{h}^\dagger \rangle^2 + \langle \hat{e}^\dagger \hat{h} \rangle^2 \\ &+ \langle \hat{e} \hat{h}^\dagger \rangle^2 + \langle \hat{f}^\dagger \hat{g} \rangle^2 + \langle \hat{f} \hat{g}^\dagger \rangle^2 - 2(\langle \hat{e}^\dagger \hat{g} \rangle \langle \hat{e} \hat{g}^\dagger \rangle \\ &+ \langle \hat{f}^\dagger \hat{h} \rangle \langle \hat{f} \hat{h}^\dagger \rangle + \langle \hat{e}^\dagger \hat{h} \rangle \langle \hat{e} \hat{h}^\dagger \rangle + \langle \hat{f}^\dagger \hat{g} \rangle \langle \hat{f} \hat{g}^\dagger \rangle + \langle \hat{e}^\dagger \hat{h} \rangle \langle \hat{f}^\dagger \hat{g} \rangle \\ &+ \langle \hat{e} \hat{h}^\dagger \rangle \langle \hat{f} \hat{g}^\dagger \rangle - \langle \hat{e}^\dagger \hat{g} \rangle \langle \hat{f}^\dagger \hat{h} \rangle - \langle \hat{e} \hat{g}^\dagger \rangle \langle \hat{f} \hat{h}^\dagger \rangle - \langle \hat{e}^\dagger \hat{g} \rangle \langle \hat{f} \hat{h}^\dagger \rangle \\ &- \langle \hat{e} \hat{g}^\dagger \rangle \langle \hat{f}^\dagger \hat{h} \rangle - \langle \hat{e}^\dagger \hat{h} \rangle \langle \hat{f} \hat{g}^\dagger \rangle - \langle \hat{e} \hat{h}^\dagger \rangle \langle \hat{f}^\dagger \hat{g} \rangle) - 2i(\langle \hat{e}^\dagger \hat{g} \rangle \langle \hat{e}^\dagger \hat{h} \rangle \\ &+ \langle \hat{e} \hat{g}^\dagger \rangle \langle \hat{e}^\dagger \hat{h} \rangle + \langle \hat{e} \hat{g}^\dagger \rangle \langle \hat{f} \hat{g}^\dagger \rangle + \langle \hat{e}^\dagger \hat{g} \rangle \langle \hat{f} \hat{g}^\dagger \rangle + \langle \hat{f} \hat{h}^\dagger \rangle \langle \hat{e} \hat{h}^\dagger \rangle \\ &+ \langle \hat{e} \hat{h}^\dagger \rangle \langle \hat{f}^\dagger \hat{h} \rangle + \langle \hat{f}^\dagger \hat{h} \rangle \langle \hat{f}^\dagger \hat{g} \rangle + \langle \hat{f} \hat{h}^\dagger \rangle \langle \hat{f}^\dagger \hat{g} \rangle - \langle \hat{e} \hat{g}^\dagger \rangle \langle \hat{e} \hat{h}^\dagger \rangle \\ &- \langle \hat{e}^\dagger \hat{g} \rangle \langle \hat{e} \hat{h}^\dagger \rangle - \langle \hat{e}^\dagger \hat{g} \rangle \langle \hat{f}^\dagger \hat{g} \rangle - \langle \hat{e} \hat{g}^\dagger \rangle \langle \hat{f}^\dagger \hat{g} \rangle - \langle \hat{f}^\dagger \hat{h} \rangle \langle \hat{e}^\dagger \hat{h} \rangle \\ &- \langle \hat{f} \hat{h}^\dagger \rangle \langle \hat{e}^\dagger \hat{h} \rangle - \langle \hat{f} \hat{h}^\dagger \rangle \langle \hat{f} \hat{g}^\dagger \rangle - \langle \hat{f}^\dagger \hat{h} \rangle \langle \hat{f} \hat{g}^\dagger \rangle) \end{aligned} \end{bmatrix},$$

where

$$\hat{Q}_1 = \hat{g}^\dagger \hat{g}^\dagger - \hat{h}^\dagger \hat{h}^\dagger - 2i \hat{g}^\dagger \hat{h}^\dagger,$$

$$\begin{aligned}
\hat{Q}_2 &= \hat{g}\hat{g}^\dagger + \hat{h}\hat{h}^\dagger + i(\hat{g}^\dagger\hat{h} - \hat{g}\hat{h}^\dagger), \\
\hat{Q}_3 &= \hat{g}^\dagger\hat{g} + \hat{h}^\dagger\hat{h} + i(\hat{g}^\dagger\hat{h} - \hat{g}\hat{h}^\dagger), \\
\hat{Q}_4 &= \hat{h}^\dagger\hat{h}^\dagger - \hat{g}^\dagger\hat{g}^\dagger - 2i\hat{g}^\dagger\hat{h}^\dagger, \\
\hat{Q}_5 &= \hat{g}\hat{g}^\dagger + \hat{h}\hat{h}^\dagger + i(\hat{g}\hat{h}^\dagger - \hat{g}^\dagger\hat{h}), \\
\hat{Q}_6 &= \hat{g}^\dagger\hat{g} + \hat{h}^\dagger\hat{h} + i(\hat{g}\hat{h}^\dagger - \hat{g}^\dagger\hat{h}), \\
\hat{Q}_7 &= 2i(\hat{g}^\dagger\hat{g}^\dagger + \hat{h}^\dagger\hat{h}^\dagger),
\end{aligned}$$

and

$$\hat{Q}_8 = 2(\hat{g}\hat{h}^\dagger + \hat{g}^\dagger\hat{h}) - i(\hat{g}\hat{g}^\dagger + \hat{g}^\dagger\hat{g}) + i(\hat{h}\hat{h}^\dagger + \hat{h}^\dagger\hat{h}).$$

IV. MEASUREMENT OPERATOR CROSS CORRELATIONS

The expressions for the measurement operator cross correlations that are required to evaluate the numerator in eq.(1) are

$$\langle \hat{D}_{\alpha;\alpha,\gamma} \rangle = 8 \left[\begin{aligned} & \{ \langle \hat{S}_1 \rangle + \langle \hat{S}_1^\dagger \rangle - \langle \hat{S}_2 \rangle - \langle \hat{S}_2^\dagger \rangle \} \cos \varphi - \\ & \{ \langle \hat{S}_3 \rangle + \langle \hat{S}_3^\dagger \rangle - \langle \hat{S}_4 \rangle - \langle \hat{S}_4^\dagger \rangle \} \sin \varphi - \\ & 2\{ (\langle \hat{\rho}_b^\dagger \hat{\rho}_b \rangle - \langle \hat{\rho}_a^\dagger \hat{\rho}_a \rangle) \cos \varphi - (\langle \hat{\rho}_a^\dagger \hat{\rho}_b \rangle + \langle \hat{\rho}_a \hat{\rho}_b^\dagger \rangle) \sin \varphi \} \cdot \\ & \{ (\langle \hat{\rho}_b^\dagger \hat{f} \rangle + i \langle \hat{\rho}_a^\dagger \hat{f} \rangle) + (\langle \hat{\rho}_b \hat{f}^\dagger \rangle - i \langle \hat{\rho}_a \hat{f}^\dagger \rangle) - \\ & (\langle \hat{\rho}_a^\dagger \hat{e} \rangle + i \langle \hat{\rho}_b^\dagger \hat{e} \rangle) e^{-i\varphi} - (\langle \hat{\rho}_a \hat{e}^\dagger \rangle - i \langle \hat{\rho}_b \hat{e}^\dagger \rangle) e^{i\varphi} \} \end{aligned} \right],$$

where

$$\begin{aligned}
\hat{S}_1 &= \left[(\hat{\rho}_b^\dagger \hat{\rho}_b \hat{\rho}_b^\dagger + \hat{\rho}_b^\dagger \hat{\rho}_b^\dagger \hat{\rho}_b - 2\hat{\rho}_a^\dagger \hat{\rho}_a \hat{\rho}_b^\dagger) + i(2\hat{\rho}_a^\dagger \hat{\rho}_b^\dagger \hat{\rho}_b - \hat{\rho}_a^\dagger \hat{\rho}_a \hat{\rho}_b^\dagger - \hat{\rho}_a^\dagger \hat{\rho}_b^\dagger \hat{\rho}_a) \right] \hat{f}, \\
\hat{S}_2 &= \left[(2\hat{\rho}_a^\dagger \hat{\rho}_b^\dagger \hat{\rho}_b - \hat{\rho}_a^\dagger \hat{\rho}_a \hat{\rho}_b^\dagger - \hat{\rho}_a^\dagger \hat{\rho}_b^\dagger \hat{\rho}_a) - i(2\hat{\rho}_a^\dagger \hat{\rho}_a \hat{\rho}_b^\dagger - \hat{\rho}_b^\dagger \hat{\rho}_b \hat{\rho}_b^\dagger - \hat{\rho}_b^\dagger \hat{\rho}_b^\dagger \hat{\rho}_b) \right] \hat{e} e^{-i\varphi}, \\
\hat{S}_3 &= \left[(2\hat{\rho}_a \hat{\rho}_b^\dagger \hat{\rho}_b^\dagger + \hat{\rho}_a^\dagger \hat{\rho}_b \hat{\rho}_b^\dagger + \hat{\rho}_a^\dagger \hat{\rho}_b^\dagger \hat{\rho}_b) + i(2\hat{\rho}_a^\dagger \hat{\rho}_a \hat{\rho}_b + \hat{\rho}_a \hat{\rho}_a^\dagger \hat{\rho}_b^\dagger + \hat{\rho}_a \hat{\rho}_a \hat{\rho}_b^\dagger) \right] \hat{f}, \\
\hat{S}_4 &= \left[(2\hat{\rho}_a^\dagger \hat{\rho}_a \hat{\rho}_b + \hat{\rho}_a \hat{\rho}_a^\dagger \hat{\rho}_b^\dagger + \hat{\rho}_a^\dagger \hat{\rho}_a \hat{\rho}_b^\dagger) + i(2\hat{\rho}_a \hat{\rho}_b^\dagger \hat{\rho}_b^\dagger + \hat{\rho}_a^\dagger \hat{\rho}_b \hat{\rho}_b^\dagger + \hat{\rho}_a^\dagger \hat{\rho}_b^\dagger \hat{\rho}_b) \right] \hat{e} e^{-i\varphi}; \\
\langle \hat{D}_{\alpha;\alpha,\varepsilon} \rangle &= 4 \left[\begin{aligned} & \{ \langle \hat{U}_1 \rangle + \langle \hat{U}_1^\dagger \rangle - \langle \hat{U}_2 \rangle - \langle \hat{U}_2^\dagger \rangle \} \cos \varphi - \\ & \{ \langle \hat{U}_3 \rangle + \langle \hat{U}_3^\dagger \rangle + \langle \hat{U}_4 \rangle + \langle \hat{U}_4^\dagger \rangle \} \sin \varphi - \\ & 2\{ (\langle \hat{\rho}_b^\dagger \hat{\rho}_b \rangle - \langle \hat{\rho}_a^\dagger \hat{\rho}_a \rangle) \cos \varphi - (\langle \hat{\rho}_a^\dagger \hat{\rho}_b \rangle + \langle \hat{\rho}_a \hat{\rho}_b^\dagger \rangle) \sin \varphi \} \cdot \\ & \{ (\langle \hat{\rho}_a^\dagger \hat{g} \rangle + \langle \hat{\rho}_b^\dagger \hat{h} \rangle) (1 - e^{-i\varphi}) + (\langle \hat{\rho}_a \hat{g}^\dagger \rangle + \langle \hat{\rho}_b \hat{h}^\dagger \rangle) (1 - e^{i\varphi}) + \\ & i(\langle \hat{\rho}_a^\dagger \hat{h} \rangle - \langle \hat{\rho}_b^\dagger \hat{g} \rangle) (1 + e^{-i\varphi}) + i(\langle \hat{\rho}_b \hat{g}^\dagger \rangle - \langle \hat{\rho}_a \hat{h}^\dagger \rangle) (1 + e^{i\varphi}) \} \end{aligned} \right],
\end{aligned}$$

where

$$\begin{aligned}
\hat{U}_1 &= \left(2\hat{\rho}_a^\dagger \hat{\rho}_b^\dagger \hat{\rho}_b - \hat{\rho}_a^\dagger \hat{\rho}_a \hat{\rho}_a^\dagger - \hat{\rho}_a^\dagger \hat{\rho}_a^\dagger \hat{\rho}_a \right) \left[\hat{g} (1 - e^{-i\varphi}) + i\hat{h} (1 + e^{-i\varphi}) \right], \\
\hat{U}_2 &= \left(2\hat{\rho}_a^\dagger \hat{\rho}_a \hat{\rho}_b^\dagger - \hat{\rho}_b^\dagger \hat{\rho}_b \hat{\rho}_b^\dagger - \hat{\rho}_b^\dagger \hat{\rho}_b^\dagger \hat{\rho}_b \right) \left[\hat{h} (1 - e^{-i\varphi}) - i\hat{g} (1 + e^{-i\varphi}) \right], \\
\hat{U}_3 &= \left(2\hat{\rho}_a^\dagger \hat{\rho}_a^\dagger \hat{\rho}_b + \hat{\rho}_a \hat{\rho}_a^\dagger \hat{\rho}_b^\dagger + \hat{\rho}_a^\dagger \hat{\rho}_a \hat{\rho}_b^\dagger \right) \left[\hat{g} (1 - e^{-i\varphi}) + i\hat{h} (1 + e^{-i\varphi}) \right], \\
\hat{U}_4 &= \left(2\hat{\rho}_a \hat{\rho}_b^\dagger \hat{\rho}_b^\dagger + \hat{\rho}_a^\dagger \hat{\rho}_b \hat{\rho}_b^\dagger + \hat{\rho}_a^\dagger \hat{\rho}_b^\dagger \hat{\rho}_b \right) \left[\hat{h} (1 - e^{-i\varphi}) - i\hat{g} (1 + e^{-i\varphi}) \right]; \\
\langle \hat{D}_{\gamma;\alpha,\gamma} \rangle &= 4 \left[\frac{\langle \hat{V}_1 \rangle + \langle \hat{V}_1^\dagger \rangle + \langle \hat{V}_2 \rangle + \langle \hat{V}_2^\dagger \rangle - \langle \hat{V}_3 \rangle - \langle \hat{V}_3^\dagger \rangle - \langle \hat{V}_4 \rangle - \langle \hat{V}_4^\dagger \rangle}{2 \left(\langle \hat{e}^\dagger \hat{f} \rangle - \langle \hat{e} \hat{f}^\dagger \rangle \right) \left[\left(\langle \hat{\rho}_a \hat{f}^\dagger \rangle + i \langle \hat{\rho}_b \hat{f}^\dagger \rangle \right) - \left(\langle \hat{\rho}_a^\dagger \hat{f} \rangle - i \langle \hat{\rho}_b^\dagger \hat{f} \rangle \right) \right] +} \right. \\
&\quad \left. \frac{\left(\langle \hat{\rho}_b^\dagger \hat{e} \rangle - i \langle \hat{\rho}_a^\dagger \hat{e} \rangle \right) e^{-i\varphi} - \left(\langle \hat{\rho}_b \hat{e}^\dagger \rangle + i \langle \hat{\rho}_a \hat{e}^\dagger \rangle \right) e^{i\varphi}}{\left(\langle \hat{\rho}_b^\dagger \hat{e} \rangle - i \langle \hat{\rho}_a^\dagger \hat{e} \rangle \right) e^{-i\varphi} - \left(\langle \hat{\rho}_b \hat{e}^\dagger \rangle + i \langle \hat{\rho}_a \hat{e}^\dagger \rangle \right) e^{i\varphi}} \right],
\end{aligned}$$

where

$$\begin{aligned}
\hat{V}_1 &= (\hat{\rho}_a + i\hat{\rho}_b) \hat{e}^\dagger \hat{f} \hat{f}^\dagger, \\
\hat{V}_2 &= (\hat{\rho}_a + i\hat{\rho}_b) \hat{e}^\dagger \hat{f}^\dagger \hat{f}, \\
\hat{V}_3 &= 2(\hat{\rho}_a + i\hat{\rho}_b) \hat{e} \hat{f}^\dagger \hat{f}^\dagger, \\
\hat{V}_4 &= (\hat{\rho}_b + i\hat{\rho}_a) \left(2\hat{e}^\dagger \hat{e}^\dagger \hat{f} - \hat{e}^\dagger \hat{e} \hat{f}^\dagger - \hat{e} \hat{e}^\dagger \hat{f}^\dagger \right) e^{i\varphi}; \\
\langle \hat{D}_{\gamma;\gamma,\varepsilon} \rangle &= 2 \left[\frac{\langle \hat{W}_1 \rangle + \langle \hat{W}_1^\dagger \rangle + \langle \hat{W}_2 \rangle + \langle \hat{W}_2^\dagger \rangle - 2 \left(\langle \hat{e}^\dagger \hat{f} \rangle - \langle \hat{e} \hat{f}^\dagger \rangle \right)}{\left[\left(\langle \hat{f}^\dagger \hat{g} \rangle - i \langle \hat{f}^\dagger \hat{h} \rangle \right) - \left(\langle \hat{f} \hat{g}^\dagger \rangle + i \langle \hat{f} \hat{h}^\dagger \rangle \right) - \right.} \right. \\
&\quad \left. \left(\langle \hat{e}^\dagger \hat{h} \rangle - i \langle \hat{e}^\dagger \hat{g} \rangle \right) + \left(\langle \hat{e} \hat{h}^\dagger \rangle + i \langle \hat{e} \hat{g}^\dagger \rangle \right) \right],
\end{aligned}$$

where

$$\begin{aligned}
\hat{W}_1 &= \left(\hat{e}^\dagger \hat{f} \hat{f}^\dagger + \hat{e}^\dagger \hat{f}^\dagger \hat{f} - 2\hat{e} \hat{f}^\dagger \hat{f}^\dagger \right) (\hat{g} - i\hat{h}), \\
\hat{W}_2 &= \left(\hat{e}^\dagger \hat{e} \hat{f}^\dagger + \hat{e} \hat{e}^\dagger \hat{f}^\dagger - 2\hat{e}^\dagger \hat{e} \hat{f} \right) (\hat{h} - i\hat{g}); \\
\langle \hat{D}_{\varepsilon;\alpha,\varepsilon} \rangle &= \left[\frac{\langle \hat{H}_1 \rangle + \langle \hat{H}_1^\dagger \rangle + \langle \hat{H}_2 \rangle + \langle \hat{H}_2^\dagger \rangle + \langle \hat{H}_3 \rangle + \langle \hat{H}_3^\dagger \rangle + \langle \hat{H}_4 \rangle + \langle \hat{H}_4^\dagger \rangle}{-2 \left[\langle \hat{g}^\dagger \hat{g} \rangle - \langle \hat{h}^\dagger \hat{h} \rangle \right] \cdot \left[\left(\langle \hat{\rho}_a^\dagger \hat{g} \rangle + \langle \hat{\rho}_b^\dagger \hat{h} \rangle \right) (1 - e^{-i\varphi}) \right.} \right. \\
&\quad \left. + \left(\langle \hat{\rho}_a \hat{g}^\dagger \rangle + \langle \hat{\rho}_b \hat{h}^\dagger \rangle \right) (1 - e^{i\varphi}) + i \left(\langle \hat{\rho}_a^\dagger \hat{h} \rangle - \langle \hat{\rho}_b^\dagger \hat{g} \rangle \right) (1 + e^{-i\varphi}) \right. \\
&\quad \left. - i \left(\langle \hat{\rho}_a \hat{h}^\dagger \rangle - \langle \hat{\rho}_b \hat{g}^\dagger \rangle \right) (1 + e^{i\varphi}) \right],
\end{aligned}$$

where

$$\begin{aligned}
\hat{H}_1 &= \hat{\rho}_b \left(2\hat{g}^\dagger \hat{g} \hat{h}^\dagger - \hat{h}^\dagger \hat{h} \hat{h}^\dagger - \hat{h}^\dagger \hat{h}^\dagger \hat{h} \right) (1 - e^{i\varphi}), \\
\hat{H}_2 &= -\hat{\rho}_a \left(2\hat{g}^\dagger \hat{h}^\dagger \hat{h} - \hat{g}^\dagger \hat{g} \hat{g}^\dagger - \hat{g}^\dagger \hat{g}^\dagger \hat{g} \right) (1 - e^{i\varphi}), \\
\hat{H}_3 &= -i\hat{\rho}_a \left(2\hat{g}^\dagger \hat{g} \hat{h}^\dagger - \hat{h}^\dagger \hat{h} \hat{h}^\dagger - \hat{h}^\dagger \hat{h}^\dagger \hat{h} \right) (1 + e^{i\varphi}),
\end{aligned}$$

$$\hat{H}_4 = -i\hat{\rho}_b \left(2\hat{g}^\dagger \hat{h}^\dagger \hat{h} - \hat{g}^\dagger \hat{g} \hat{g}^\dagger - \hat{g}^\dagger \hat{g}^\dagger \hat{g} \right) (1 + e^{i\varphi});$$

$$\langle \hat{D}_{\varepsilon;\gamma,\varepsilon} \rangle = \begin{bmatrix} \langle \hat{L}_1 \rangle + \langle \hat{L}_1^\dagger \rangle + \langle \hat{L}_2 \rangle + \langle \hat{L}_2^\dagger \rangle + \langle \hat{L}_3 \rangle + \langle \hat{L}_3^\dagger \rangle + \langle \hat{L}_4 \rangle + \langle \hat{L}_4^\dagger \rangle \\ -2 \left(\langle \hat{g}^\dagger \hat{g} \rangle - \langle \hat{h}^\dagger \hat{h} \rangle \right) [(\langle \hat{e}^\dagger \hat{g} \rangle + \langle \hat{e} \hat{g}^\dagger \rangle) - (\langle \hat{f}^\dagger \hat{h} \rangle + \langle \hat{f} \hat{h}^\dagger \rangle)] \\ + i \left(\langle \hat{e}^\dagger \hat{h} \rangle - \langle \hat{e} \hat{h}^\dagger \rangle \right) - i \left(\langle \hat{f}^\dagger \hat{g} \rangle - \langle \hat{f} \hat{g}^\dagger \rangle \right) \end{bmatrix},$$

where

$$\begin{aligned} \hat{L}_1 &= -\hat{e} \left(2\hat{g}^\dagger \hat{h}^\dagger \hat{h} - \hat{g}^\dagger \hat{g} \hat{g}^\dagger - \hat{g}^\dagger \hat{g}^\dagger \hat{g} \right), \\ \hat{L}_2 &= -\hat{f} \left(2\hat{g}^\dagger \hat{g} \hat{h}^\dagger - \hat{h}^\dagger \hat{h} \hat{h}^\dagger - \hat{h}^\dagger \hat{h}^\dagger \hat{h} \right), \\ \hat{L}_3 &= -i\hat{e} \left(2\hat{g}^\dagger \hat{g} \hat{h}^\dagger - \hat{h}^\dagger \hat{h} \hat{h}^\dagger - \hat{h}^\dagger \hat{h}^\dagger \hat{h} \right), \\ \hat{L}_4 &= -i\hat{f} \left(2\hat{g}^\dagger \hat{h}^\dagger \hat{h} - \hat{g}^\dagger \hat{g} \hat{g}^\dagger - \hat{g}^\dagger \hat{g}^\dagger \hat{g} \right); \end{aligned}$$

$$\langle \hat{D}_{\alpha,\gamma;\alpha,\varepsilon} \rangle = 2 \begin{bmatrix} \langle \hat{K}_1 \rangle + \langle \hat{K}_1^\dagger \rangle + \langle \hat{K}_2 \rangle + \langle \hat{K}_2^\dagger \rangle - \langle \hat{K}_3 \rangle - \langle \hat{K}_3^\dagger \rangle - \\ \langle \hat{K}_4 \rangle - \langle \hat{K}_4^\dagger \rangle - \langle \hat{K}_5 \rangle - \langle \hat{K}_5^\dagger \rangle - \langle \hat{K}_6 \rangle - \langle \hat{K}_6^\dagger \rangle - \\ 2[\langle \hat{\rho}_b^\dagger \hat{f} \rangle + \langle \hat{\rho}_b \hat{f}^\dagger \rangle - \langle \hat{\rho}_a^\dagger \hat{e} \rangle e^{-i\varphi} - \langle \hat{\rho}_a \hat{e}^\dagger \rangle e^{i\varphi} + \\ i \langle \hat{\rho}_a^\dagger \hat{f} \rangle - i \langle \hat{\rho}_a \hat{f}^\dagger \rangle - i \langle \hat{\rho}_b^\dagger \hat{e} \rangle e^{-i\varphi} + i \langle \hat{\rho}_b \hat{e}^\dagger \rangle e^{i\varphi}] \cdot \\ [(\langle \hat{\rho}_a^\dagger \hat{g} \rangle + \langle \hat{\rho}_b^\dagger \hat{h} \rangle) (1 - e^{-i\varphi}) + (\langle \hat{\rho}_a \hat{g}^\dagger \rangle + \langle \hat{\rho}_b \hat{h}^\dagger \rangle) (1 - e^{i\varphi}) + \\ i (\langle \hat{\rho}_a^\dagger \hat{h} \rangle - \langle \hat{\rho}_b^\dagger \hat{g} \rangle) (1 + e^{-i\varphi}) + i (\langle \hat{\rho}_b \hat{g}^\dagger \rangle - \langle \hat{\rho}_a \hat{h}^\dagger \rangle) (1 + e^{i\varphi})] \end{bmatrix},$$

where

$$\begin{aligned} \hat{K}_1 &= \begin{bmatrix} 2\hat{\rho}_b^\dagger \hat{\rho}_b^\dagger \hat{f} \hat{h} + 2\hat{\rho}_a^\dagger \hat{\rho}_b^\dagger \hat{f} \hat{g} + 2\hat{\rho}_a^\dagger \hat{\rho}_b \hat{f}^\dagger \hat{g} + \hat{\rho}_b \hat{\rho}_b^\dagger \hat{f}^\dagger \hat{h} + \hat{\rho}_b^\dagger \hat{\rho}_b \hat{f}^\dagger \hat{h} + \\ i(2\hat{\rho}_a^\dagger \hat{\rho}_a^\dagger \hat{f} \hat{g} - \hat{\rho}_a \hat{\rho}_a^\dagger \hat{f}^\dagger \hat{g} - \hat{\rho}_a^\dagger \hat{\rho}_a \hat{f}^\dagger \hat{g} + 2\hat{\rho}_a^\dagger \hat{\rho}_b \hat{f} \hat{h} - 2\hat{\rho}_a \hat{\rho}_b^\dagger \hat{f}^\dagger \hat{h}) \end{bmatrix} (1 - e^{-i\varphi}), \\ \hat{K}_2 &= 2 \left[\hat{\rho}_a^\dagger \hat{\rho}_b^\dagger \hat{e} \hat{h} - \hat{\rho}_b^\dagger \hat{\rho}_b^\dagger \hat{e} \hat{g} - i(\hat{\rho}_a^\dagger \hat{\rho}_a^\dagger \hat{e} \hat{h} - \hat{\rho}_a^\dagger \hat{\rho}_b^\dagger \hat{e} \hat{g}) \right] (1 + e^{-i\varphi}) e^{-i\varphi}, \\ \hat{K}_3 &= \begin{bmatrix} 2\hat{\rho}_a^\dagger \hat{\rho}_a^\dagger \hat{f} \hat{h} + \hat{\rho}_a \hat{\rho}_a^\dagger \hat{f}^\dagger \hat{h} + \hat{\rho}_a^\dagger \hat{\rho}_a \hat{f}^\dagger \hat{h} - 2\hat{\rho}_a^\dagger \hat{\rho}_b^\dagger \hat{f} \hat{g} + 2\hat{\rho}_a \hat{\rho}_b^\dagger \hat{f}^\dagger \hat{g} - \\ i(2\hat{\rho}_a^\dagger \hat{\rho}_b^\dagger \hat{f} \hat{h} + 2\hat{\rho}_a^\dagger \hat{\rho}_b \hat{f}^\dagger \hat{h} - 2\hat{\rho}_b^\dagger \hat{\rho}_b^\dagger \hat{f} \hat{g} - \hat{\rho}_b \hat{\rho}_b^\dagger \hat{f}^\dagger \hat{g} - \hat{\rho}_b^\dagger \hat{\rho}_b \hat{f}^\dagger \hat{g}) \end{bmatrix} (1 + e^{-i\varphi}), \\ \hat{K}_4 &= 2 \left[\hat{\rho}_a \hat{\rho}_a^\dagger \hat{e}^\dagger \hat{g}^\dagger + \hat{\rho}_a \hat{\rho}_b \hat{e}^\dagger \hat{h}^\dagger - i(\hat{\rho}_a \hat{\rho}_b \hat{e}^\dagger \hat{g}^\dagger + \hat{\rho}_b \hat{\rho}_b \hat{e}^\dagger \hat{h}^\dagger) \right] (1 - e^{i\varphi}) e^{i\varphi}, \\ \hat{K}_5 &= \left[2\hat{\rho}_a \hat{\rho}_b^\dagger \hat{e} \hat{h}^\dagger - \hat{\rho}_b^\dagger \hat{\rho}_b \hat{e} \hat{g}^\dagger - \hat{\rho}_b \hat{\rho}_b^\dagger \hat{e} \hat{g}^\dagger - i(\hat{\rho}_a^\dagger \hat{\rho}_a \hat{e} \hat{h}^\dagger + \hat{\rho}_a \hat{\rho}_a^\dagger \hat{e} \hat{h}^\dagger - 2\hat{\rho}_a^\dagger \hat{\rho}_b \hat{e} \hat{g}^\dagger) \right] (1 + e^{i\varphi}) e^{-i\varphi}, \\ \hat{K}_6 &= \left[2\hat{\rho}_a^\dagger \hat{\rho}_b \hat{e} \hat{h}^\dagger + \hat{\rho}_a^\dagger \hat{\rho}_a \hat{e} \hat{g}^\dagger + \hat{\rho}_a \hat{\rho}_a^\dagger \hat{e} \hat{g}^\dagger + i(2\hat{\rho}_a \hat{\rho}_b^\dagger \hat{e} \hat{g}^\dagger + \hat{\rho}_b^\dagger \hat{\rho}_b \hat{e} \hat{h}^\dagger + \hat{\rho}_b \hat{\rho}_b^\dagger \hat{e} \hat{h}^\dagger) \right] (1 - e^{i\varphi}) e^{-i\varphi}; \end{aligned}$$

$$\langle \hat{D}_{\alpha,\gamma;\gamma,\varepsilon} \rangle = 2 \left[\begin{array}{c} \langle \hat{O}_1 \rangle + \langle \hat{O}_1^\dagger \rangle - \langle \hat{O}_2 \rangle - \langle \hat{O}_2^\dagger \rangle - \langle \hat{O}_3 \rangle - \langle \hat{O}_3^\dagger \rangle - \\ \langle \hat{O}_4 \rangle - \langle \hat{O}_4^\dagger \rangle - \langle \hat{O}_5 \rangle - \langle \hat{O}_5^\dagger \rangle - \langle \hat{O}_6 \rangle - \langle \hat{O}_6^\dagger \rangle - \\ 2\{\langle \hat{\rho}_a^\dagger \hat{f} \rangle + \langle \hat{\rho}_b \hat{f}^\dagger \rangle - \langle \hat{\rho}_a^\dagger \hat{e} \rangle e^{-i\varphi} - \langle \hat{\rho}_a \hat{e}^\dagger \rangle e^{i\varphi} + \\ i[\langle \hat{\rho}_a^\dagger \hat{f} \rangle - \langle \hat{\rho}_a \hat{f}^\dagger \rangle - \langle \hat{\rho}_b^\dagger \hat{e} \rangle e^{-i\varphi} + \langle \hat{\rho}_b \hat{e}^\dagger \rangle e^{i\varphi}] \}. \\ \{ \langle \hat{e}^\dagger \hat{g} \rangle + \langle \hat{e} \hat{g}^\dagger \rangle - \langle \hat{f}^\dagger \hat{h} \rangle - \langle \hat{f} \hat{h}^\dagger \rangle + i[\langle \hat{e}^\dagger \hat{h} \rangle - \langle \hat{e} \hat{h}^\dagger \rangle - \langle \hat{f}^\dagger \hat{g} \rangle + \langle \hat{f} \hat{g}^\dagger \rangle] \} \end{array} \right],$$

where

$$\begin{aligned} \hat{O}_1 &= 2 \left\{ \begin{array}{l} \hat{\rho}_b^\dagger [\hat{e}^\dagger \hat{f} \hat{g} + \hat{e} \hat{f} \hat{g}^\dagger - \hat{e} \hat{f}^\dagger \hat{g} e^{-i\varphi} + \hat{e} \hat{f} \hat{g}^\dagger e^{-i\varphi}] + \\ i \hat{\rho}_a^\dagger [\hat{e}^\dagger \hat{f} \hat{g} + \hat{e} \hat{f} \hat{g}^\dagger + \hat{e} \hat{f}^\dagger \hat{g} e^{-i\varphi} - \hat{e} \hat{f} \hat{g}^\dagger e^{-i\varphi}] \end{array} \right\}, \\ \hat{O}_2 &= (\hat{\rho}_b^\dagger + i \hat{\rho}_a^\dagger) (2 \hat{f} \hat{f} \hat{h}^\dagger + \hat{f} \hat{f}^\dagger \hat{h} + \hat{f}^\dagger \hat{f} \hat{h}), \\ \hat{O}_3 &= 2 \left\{ \begin{array}{l} \hat{\rho}_a^\dagger [\hat{e}^\dagger \hat{f} \hat{h} - \hat{e} \hat{f} \hat{h}^\dagger - \hat{e} \hat{f}^\dagger \hat{h} e^{-i\varphi} - \hat{e} \hat{f} \hat{h}^\dagger e^{-i\varphi}] - \\ i \hat{\rho}_b^\dagger [\hat{e}^\dagger \hat{f} \hat{h} - \hat{e} \hat{f} \hat{h}^\dagger + \hat{e} \hat{f}^\dagger \hat{h} e^{-i\varphi} + \hat{e} \hat{f} \hat{h}^\dagger e^{-i\varphi}] \end{array} \right\}, \\ \hat{O}_4 &= (\hat{\rho}_a^\dagger - i \hat{\rho}_b^\dagger) (2 \hat{f} \hat{f} \hat{g}^\dagger - \hat{f} \hat{f}^\dagger \hat{g} - \hat{f}^\dagger \hat{f} \hat{g}), \\ \hat{O}_5 &= (\hat{\rho}_b^\dagger - i \hat{\rho}_a^\dagger) (2 \hat{e} \hat{e} \hat{h}^\dagger - \hat{e} \hat{e}^\dagger \hat{h} - \hat{e}^\dagger \hat{e} \hat{h}) e^{-i\varphi}, \\ \hat{O}_6 &= (\hat{\rho}_a^\dagger + i \hat{\rho}_b^\dagger) (2 \hat{e} \hat{e} \hat{g}^\dagger + \hat{e} \hat{e}^\dagger \hat{g} + \hat{e}^\dagger \hat{e} \hat{g}) e^{-i\varphi}; \end{aligned}$$

and

$$\langle \hat{D}_{\alpha,\varepsilon;\gamma,\varepsilon} \rangle = \left[\begin{array}{c} \langle \hat{M}_1 \rangle + \langle \hat{M}_1^\dagger \rangle + \langle \hat{M}_2 \rangle + \langle \hat{M}_2^\dagger \rangle \\ -2\{(\langle \hat{\rho}_a^\dagger \hat{g} \rangle + \langle \hat{\rho}_b \hat{h}^\dagger \rangle)(1 - e^{-i\varphi}) + (\langle \hat{\rho}_a \hat{g}^\dagger \rangle + \langle \hat{\rho}_b \hat{h}^\dagger \rangle)(1 - e^{i\varphi}) \\ + i(\langle \hat{\rho}_a^\dagger \hat{h} \rangle - \langle \hat{\rho}_b \hat{g}^\dagger \rangle)(1 + e^{-i\varphi}) - i(\langle \hat{\rho}_a \hat{h}^\dagger \rangle - \langle \hat{\rho}_b \hat{g}^\dagger \rangle)(1 + e^{i\varphi}) \}. \\ \{(\langle \hat{e}^\dagger \hat{g} \rangle + \langle \hat{e} \hat{g}^\dagger \rangle - \langle \hat{f}^\dagger \hat{h} \rangle - \langle \hat{f} \hat{h}^\dagger \rangle) + i(\langle \hat{e}^\dagger \hat{h} \rangle - \langle \hat{e} \hat{h}^\dagger \rangle - \langle \hat{f}^\dagger \hat{g} \rangle + \langle \hat{f} \hat{g}^\dagger \rangle)\} \end{array} \right],$$

where

$$\begin{aligned} \hat{M}_1 &= \left\{ \begin{array}{l} [\hat{\rho}_a^\dagger (2 \hat{e}^\dagger \hat{g} \hat{g} + \hat{e} \hat{g} \hat{g}^\dagger + \hat{e} \hat{g}^\dagger \hat{g} - 2 \hat{f}^\dagger \hat{g} \hat{h} - 2 \hat{f} \hat{g} \hat{h}^\dagger) + \\ \hat{\rho}_b^\dagger (2 \hat{e}^\dagger \hat{g} \hat{h} + 2 \hat{e} \hat{g}^\dagger \hat{h} - 2 \hat{f}^\dagger \hat{h} \hat{h} - \hat{f} \hat{h} \hat{h}^\dagger - \hat{f} \hat{h}^\dagger \hat{h}) + \\ i[\hat{\rho}_a^\dagger (2 \hat{e}^\dagger \hat{g} \hat{h} - 2 \hat{e} \hat{h}^\dagger \hat{g} - 2 \hat{f}^\dagger \hat{g} \hat{g} + \hat{f} \hat{g} \hat{g}^\dagger + \hat{f} \hat{g}^\dagger \hat{g}) + \\ \hat{\rho}_b^\dagger (2 \hat{e}^\dagger \hat{h} \hat{h} - \hat{e} \hat{h} \hat{h}^\dagger - \hat{e} \hat{h}^\dagger \hat{h} - 2 \hat{f}^\dagger \hat{g} \hat{h} + 2 \hat{f} \hat{g}^\dagger \hat{h})] \end{array} \right\} (1 - e^{-i\varphi}), \\ \hat{M}_2 &= \left\{ \begin{array}{l} [\hat{\rho}_a^\dagger (2 \hat{f}^\dagger \hat{g} \hat{h} - 2 \hat{f} \hat{g}^\dagger \hat{h} - 2 \hat{e}^\dagger \hat{h} \hat{h} + \hat{e} \hat{h} \hat{h}^\dagger + \hat{e} \hat{h}^\dagger \hat{h}) + \\ \hat{\rho}_b^\dagger (2 \hat{e}^\dagger \hat{g} \hat{h} - 2 \hat{e} \hat{g} \hat{h}^\dagger - 2 \hat{f}^\dagger \hat{g} \hat{g} + \hat{f} \hat{g} \hat{g}^\dagger + \hat{f} \hat{g}^\dagger \hat{g}) + \\ i[\hat{\rho}_a^\dagger (2 \hat{e}^\dagger \hat{g} \hat{h} + 2 \hat{e} \hat{g}^\dagger \hat{h} - 2 \hat{f}^\dagger \hat{h} \hat{h} - \hat{f} \hat{h} \hat{h}^\dagger - \hat{f} \hat{h}^\dagger \hat{h}) + \\ \hat{\rho}_b^\dagger (2 \hat{f}^\dagger \hat{g} \hat{h} + 2 \hat{f} \hat{g} \hat{h}^\dagger - 2 \hat{e}^\dagger \hat{g} \hat{g} - \hat{e} \hat{g} \hat{g}^\dagger - \hat{e} \hat{g}^\dagger \hat{g})] \end{array} \right\} (1 + e^{-i\varphi}). \end{aligned}$$

V. APPLICATIONS

For the purpose of validation, it is demonstrated in this section that when the expressions for the loss model given in the previous three sections are used in conjunction with eq.(1), they yield the required phase sensitivity results that have been previously developed in PI for the lossless MZI and the ground state MZI . As an additional illustrative application, this model is also used to derive the phase sensitivity $\Delta^2\varphi_\gamma$ for an MZI with its two internal arms (the " γ region") in the excited state $|\psi_e\rangle|\psi_f\rangle = |1\rangle|1\rangle$ and the phase sensitivity $\Delta^2\varphi_\varepsilon$ for an MZI with its two channels between the output ports and ideal detectors (the " ε region") in the excited state $|\psi_g\rangle|\psi_h\rangle = |1\rangle|1\rangle$. These results are used to determine a condition which relates the expected number of photons entering an MZI to its efficiency parameters, such that $\Delta^2\varphi_\gamma > \Delta^2\varphi_\varepsilon$ when this condition is satisfied.

A. Phase sensitivity for a lossless MZI

Although this case is trivial and only requires the evaluation of eqs.(5) and (6), it is included here for completeness. When the MZI is lossless, then each of the regional efficiencies has unit value so that - with the exception $\kappa_\alpha = \frac{1}{4}$ - all of the powers and products of subscripted κ 's in eq.(1) are zero valued, $\hat{\rho}_a = \hat{a}_{in}$, and $\hat{\rho}_b = \hat{b}_{in}$. In this case eqs.(2) and (6) reduce to the quantities $\langle \hat{C}_\alpha^{(0)} \rangle$ and $\Delta^2 C_\alpha^{(0)}$ defined in section IV of PI and eq.(1) reduces to the expression for the phase sensitivity $\Delta^2\varphi_{lossless}$ for a lossless MZI given by eq.(10) therein.

B. Phase sensitivity for the ground state MZI

The ground state MZI is defined when the regional efficiency parameters in the MZI model have their values in the open real interval (0,1) and the system state is $|\Psi_{gs}\rangle = |\psi_{a_{in},b_{in}}\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle|0\rangle$. In this case, eventhough each of the subscripted κ 's in eq.(1) is non-vanishing - with the exception of the first, fourth, and fifth terms in the numerator and the first term in the denominator - each term in eq.(1) is zero when its value is determined using $|\Psi\rangle = |\Psi_{gs}\rangle$. Consequently, eq.(1) reduces to the expression for the ground state MZI phase sensitivity $\Delta^2\varphi_{gs}$ given by eq.(11) in PI.

These terms are zero because they are sums of terms which vanish due to the fact that

each contains a factor $\langle \widehat{X} \rangle_{vac} = 0$, where \widehat{X} is either a single environmental operator or a juxtaposition of several like environmental annihilation or creation operators and the subscript "vac" refers to the fact that the mean value is evaluated using only the associated environmental vacuum state. For example, $\Delta^2 C_\gamma = 0$ because each term in eq.(7) vanishes, i.e. for the first term

$$\begin{aligned}
\langle \widehat{e}^\dagger \widehat{e} \widehat{f} \widehat{f}^\dagger \rangle &= \langle \Psi_{gs} | \widehat{e}^\dagger \widehat{e} \widehat{f} \widehat{f}^\dagger | \Psi_{gs} \rangle \\
&= \langle \psi_{a_{in}, b_{in}} | \psi_{a_{in}, b_{in}} \rangle \langle \psi_c | \psi_c \rangle \langle \psi_d | \psi_d \rangle \langle \psi_e | \widehat{e}^\dagger \widehat{e} | \psi_e \rangle \langle \psi_f | \widehat{f} \widehat{f}^\dagger | \psi_f \rangle \langle \psi_g | \psi_g \rangle \langle \psi_h | \psi_h \rangle \\
&= \langle \psi_e | \widehat{e}^\dagger \widehat{e} | \psi_e \rangle \langle \psi_f | \widehat{f} \widehat{f}^\dagger | \psi_f \rangle \\
&= \langle 0 | \widehat{e}^\dagger \widehat{e} | 0 \rangle \langle 0 | \widehat{f} \widehat{f}^\dagger | 0 \rangle \\
&= 0 \cdot \langle 0 | \widehat{f} \widehat{f}^\dagger | 0 \rangle \\
&= 0
\end{aligned}$$

(the reader will also recognize $\widehat{e}^\dagger \widehat{e} = \widehat{N}$ as the number operator \widehat{N} with the property $\langle 0 | \widehat{N} | 0 \rangle = 0$), and similarly for terms two through seven. It is also easily verified that $\Delta^2 C_\alpha, \Delta^2 C_{\alpha, \gamma} \left(= 4 \langle \Psi_{gs} | [\widehat{F}_3 + \widehat{F}_6] | \Psi_{gs} \rangle \right), \Delta^2 C_{\alpha, \varepsilon} \left(= \langle \Psi_{gs} | [\widehat{R}_3 + \widehat{R}_6] | \Psi_{gs} \rangle \right)$, and $\frac{\partial \langle \widehat{C}_\alpha \rangle}{\partial \varphi}$ are as specified in section V of PI.

C. Phase sensitivities for simple excited state MZIs : A sensitivity trade-off condition

Consider now the phase sensitivities $\Delta^2 \varphi_\gamma$ and $\Delta^2 \varphi_\varepsilon$ for MZIs in the system states $|\Psi_\gamma\rangle = |\psi_{a_{in}, b_{in}}\rangle |0\rangle |0\rangle |1\rangle |1\rangle |0\rangle |0\rangle$ and $|\Psi_\varepsilon\rangle = |\psi_{a_{in}, b_{in}}\rangle |0\rangle |0\rangle |0\rangle |0\rangle |1\rangle |1\rangle$, respectively. When $|\Psi_\gamma\rangle$ is used as the model's system state it is found that all but the first, second, fourth, fifth, and sixth terms in the numerator and the first term in the denominator of eq.(1) vanish. Using eq.(8) it is also found that

$$\Delta^2 C_{\alpha, \gamma} = \Delta^2 C_{\alpha, \gamma}^{gs} + \Delta^2 C_{\alpha, \gamma}^\gamma,$$

where

$$\Delta^2 C_{\alpha, \gamma}^{gs} \equiv 4 \langle \Psi_\gamma | [\widehat{F}_3 + \widehat{F}_6] | \Psi_\gamma \rangle = 4 \langle \Psi_{gs} | [\widehat{F}_3 + \widehat{F}_6] | \Psi_{gs} \rangle$$

and

$$\Delta^2 C_{\alpha, \gamma}^\gamma \equiv 4 \langle \Psi_\gamma | [\widehat{F}_2 + \widehat{F}_3 + \widehat{F}_5 + \widehat{F}_6] | \Psi_\gamma \rangle = 4 \left[\alpha \left(\langle \widehat{a}_{in}^\dagger \widehat{a}_{in} \rangle + \langle \widehat{b}_{in}^\dagger \widehat{b}_{in} \rangle \right) + 1 \right].$$

Since each of the expressions for $\Delta^2 C_\alpha$, $\Delta^2 C_{\alpha,\varepsilon}$, and $\frac{\partial \langle \hat{C}_\alpha \rangle}{\partial \varphi}$ remains invariant when the system states $|\Psi_{gs}\rangle$ and $|\Psi_\gamma\rangle$ are used for their valuations, then

$$\Delta^2 \varphi_\gamma = \Delta^2 \varphi_{gs} + \Delta^2 \varphi_\gamma^e,$$

where

$$\Delta^2 \varphi_{gs} \equiv \frac{\kappa_\alpha^2 \Delta^2 C_\alpha + \kappa_{\alpha,\gamma}^2 \Delta^2 C_{\alpha,\gamma}^{gs} + \kappa_{\alpha,\varepsilon}^2 \Delta^2 C_{\alpha,\varepsilon}}{\left| \kappa_\alpha \frac{\partial \langle \hat{C}_\alpha \rangle}{\partial \varphi} \right|^2}$$

is the phase sensitivity for the ground state MZI and

$$\Delta^2 \varphi_\gamma^e \equiv \frac{\kappa_\gamma^2 \Delta^2 C_\gamma + \kappa_{\alpha,\gamma}^2 \Delta^2 C_{\alpha,\gamma}^\gamma + \kappa_{\gamma,\varepsilon}^2 \Delta^2 C_{\gamma,\varepsilon}}{\left| \kappa_\alpha \frac{\partial \langle \hat{C}_\alpha \rangle}{\partial \varphi} \right|^2}.$$

Evaluation of the right hand side of the last equation yields (this corrects the expression for $\Delta^2 \varphi_{es}$ given in section VI of PI)

$$\Delta^2 \varphi_\gamma = \Delta^2 \varphi_{gs} + \left[\frac{2(1-\gamma)}{\alpha^2 \gamma^2 \varepsilon} \right] \left\{ \frac{\varepsilon \left[\alpha \gamma \left(\langle \hat{a}_{in}^\dagger \hat{a}_{in} \rangle + \langle \hat{b}_{in}^\dagger \hat{b}_{in} \rangle \right) + (1-\gamma) \right] + 1}{\left[\left(\langle \hat{b}_{in}^\dagger \hat{b}_{in} \rangle - \langle \hat{a}_{in}^\dagger \hat{a}_{in} \rangle \right) \sin \varphi + \left(\langle \hat{a}_{in}^\dagger \hat{b}_{in} \rangle + \langle \hat{b}_{in}^\dagger \hat{a}_{in} \rangle \right) \cos \varphi \right]^2} \right\}. \quad (10)$$

Thus, if the " γ region" is in the excited state $|1\rangle|1\rangle$ and $\gamma \in (0, 1)$, then $\Delta^2 \varphi_\gamma > \Delta^2 \varphi_{gs}$. Observe that when there are no losses in the " γ region", i.e. when $\gamma = 1$, then - as required - $\Delta^2 \varphi_\gamma^e = 0$ so that $\Delta^2 \varphi_\gamma = \Delta^2 \varphi_{gs}$.

When $|\Psi_\varepsilon\rangle$ is used as the model's system state, then all terms in eq.(1) vanish except the first term in the denominator and the first, fourth, fifth, and sixth terms in the numerator. Also, it is determined from eq.(9) that

$$\Delta^2 C_{\alpha,\varepsilon} = \Delta^2 C_{\alpha,\varepsilon}^{gs} + \Delta^2 C_{\alpha,\varepsilon}^\varepsilon,$$

where

$$\Delta^2 C_{\alpha,\varepsilon}^{gs} \equiv \langle \Psi_\varepsilon | [\hat{R}_3 + \hat{R}_6] | \Psi_\varepsilon \rangle = \langle \Psi_{gs} | [\hat{R}_3 + \hat{R}_6] | \Psi_{gs} \rangle$$

and

$$\Delta^2 C_{\alpha,\varepsilon}^\varepsilon \equiv \langle \Psi_\varepsilon | [\hat{R}_2 + \hat{R}_3 + \hat{R}_5 + \hat{R}_6] | \Psi_\varepsilon \rangle = 8 \left[\alpha \left(\langle \hat{a}_{in}^\dagger \hat{a}_{in} \rangle + \langle \hat{b}_{in}^\dagger \hat{b}_{in} \rangle \right) + 1 \right],$$

and it is easily verified that the quantities $\Delta^2 C_\alpha$, $\Delta^2 C_{\alpha,\gamma}$, and $\frac{\partial \langle \hat{C}_\alpha \rangle}{\partial \varphi}$ yield identical expressions when the system states $|\Psi_{gs}\rangle$ and $|\Psi_\varepsilon\rangle$ are used to evaluate them. Thus,

$$\Delta^2 \varphi_\varepsilon = \Delta^2 \varphi_{gs} + \Delta^2 \varphi_\varepsilon^e,$$

where

$$\Delta^2\varphi_{gs} \equiv \frac{\kappa_\alpha^2 \Delta^2 C_\alpha + \kappa_{\alpha,\gamma} \Delta^2 C_{\alpha,\gamma} + \kappa_{\alpha,\varepsilon}^2 \Delta^2 C_{\alpha,\varepsilon}^{gs}}{\left| \kappa_\alpha \frac{\partial \langle \hat{C}_\alpha \rangle}{\partial \varphi} \right|^2}$$

is the phase sensitivity for the ground state MZI and

$$\Delta^2\varphi_\varepsilon^e \equiv \frac{\kappa_{\alpha,\varepsilon}^2 \Delta^2 C_{\alpha,\varepsilon}^e + \kappa_{\gamma,\varepsilon}^2 \Delta^2 C_{\gamma,\varepsilon}}{\left| \kappa_\alpha \frac{\partial \langle \hat{C}_\alpha \rangle}{\partial \varphi} \right|^2},$$

so that

$$\Delta^2\varphi_\varepsilon = \Delta^2\varphi_{gs} + \left[\frac{2(1-\varepsilon)}{\alpha^2 \gamma^2 \varepsilon} \right] \left\{ \frac{\alpha \gamma \left(\langle \hat{a}_{in}^\dagger \hat{a}_{in} \rangle + \langle \hat{b}_{in}^\dagger \hat{b}_{in} \rangle \right) + 1}{\left[\left(\langle \hat{b}_{in}^\dagger \hat{b}_{in} \rangle - \langle \hat{a}_{in}^\dagger \hat{a}_{in} \rangle \right) \sin \varphi + \left(\langle \hat{a}_{in}^\dagger \hat{b}_{in} \rangle + \langle \hat{b}_{in}^\dagger \hat{a}_{in} \rangle \right) \cos \varphi \right]^2} \right\}. \quad (11)$$

It is clear from this expression that when the " ε region" is in the excited state $|1\rangle|1\rangle$ and $\varepsilon \in (0, 1)$, then $\Delta^2\varphi_\varepsilon > \Delta^2\varphi_{gs}$ and when there are no losses in the " ε region" so that $\varepsilon = 1$, then - as required - $\Delta^2\varphi_\varepsilon^e = 0$ and $\Delta^2\varphi_\varepsilon = \Delta^2\varphi_{gs}$.

For the sake of further illustrating the utility of the model, note that eqs.(10) and (11) yield the difference

$$\Delta^2\varphi_\gamma - \Delta^2\varphi_\varepsilon = \frac{2\alpha\gamma[\varepsilon(1-\gamma) - (1-\varepsilon)] \left(\langle \hat{a}_{in}^\dagger \hat{a}_{in} \rangle + \langle \hat{b}_{in}^\dagger \hat{b}_{in} \rangle \right) + 2[\varepsilon(1-\gamma)^2 + \varepsilon - \gamma]}{\alpha^2 \gamma^2 \varepsilon \left[\left(\langle \hat{b}_{in}^\dagger \hat{b}_{in} \rangle - \langle \hat{a}_{in}^\dagger \hat{a}_{in} \rangle \right) \sin \varphi + \left(\langle \hat{a}_{in}^\dagger \hat{b}_{in} \rangle + \langle \hat{b}_{in}^\dagger \hat{a}_{in} \rangle \right) \cos \varphi \right]^2}.$$

Thus, for $\gamma \neq 1$ or $\varepsilon \neq 1$ it can be concluded that when the condition

$$\alpha \left(\langle \hat{a}_{in}^\dagger \hat{a}_{in} \rangle + \langle \hat{b}_{in}^\dagger \hat{b}_{in} \rangle \right) > \frac{[\gamma - \varepsilon - \varepsilon(1-\gamma)^2]}{\gamma[\varepsilon(1-\gamma) - (1-\varepsilon)]} \quad (12)$$

is satisfied, then $\Delta^2\varphi_\gamma > \Delta^2\varphi_\varepsilon$, i.e. *phase sensitivity is more degraded when an MZI is in state $|\Psi_\gamma\rangle$ than when it is in state $|\Psi_\varepsilon\rangle$* . This is an intuitively pleasing conclusion, since - unlike the case for the excited " ε region" - the degradation in sensitivity due to the excited " γ region" would be expected to experience additional degradation induced by the (unexcited) " ε region". As a special case of this result, observe that if $\gamma = \varepsilon \neq 1$, then - since the right hand side of (12) has unit value - the relationship $\Delta^2\varphi_\gamma > \Delta^2\varphi_\varepsilon$ also prevails when

$$\langle \hat{a}_{in}^\dagger \hat{a}_{in} \rangle + \langle \hat{b}_{in}^\dagger \hat{b}_{in} \rangle > \frac{1}{\alpha}.$$

VI. CLOSING REMARKS

The above provides a generalized analytical tripartite loss model for MZI phase sensitivity which is valid for both arbitrary photon input states and arbitrary system environmental states. This model subsumes the phase sensitivity models for the lossless MZI and the ground state MZI and is useful for developing specialized models for estimating the phase sensitivity, as well as for performing associated design trade-off analyses, for MZIs and MZI-like instruments which operate in environmental regimes which are not contained within the envelope of validity for the ground state model.

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- [1] A. D. Parks, S. E. Spence, J. E. Troupe, and N. J. Rodecap, Rev. Sci. Instrum. **76**, 043103 (2005).